Section 4.9: Implicit Differentiation

Goal

In this section, we will learn how to to find $\frac{dy}{dx}$ when y is defined implicitly in terms of x.

Implicit versus explicit

Bob shows up at work one day and his boss says, "Bob, your work is garbage; don't bother showing up tomorrow." Bob's boss didn't explicitly say, "you're fired," but it was implicit.

Definition 1.

1. An equation of the form $y = f(x)$ (i.e. $y =$ junk with x's and no y's) is said to define y *explicitly* as a function of x.
2. An equation in x and y defines a function f *implicitly* (possibly more than one) if the graph of $y = f(x)$ coincides with a portion of the graph of the equation.

Example 2. Implicit or explicit?

(a) $xy = 1$

(b) $x^2 + y^2 = 1$

(c) $x^3 + y^3 = 3xy$*

Question 3. How do we find $\frac{dy}{dx}$ for implicit functions?

*The graph of this equation is called the *Folium of Descartes*. It was proposed as a counterexample to Fermat's extremum finding techniques. "Hey Fermat, find tangent lines on this"; which he did.

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A naive approach

One possibility would be to solve for y and then take derivative as usual.

Example 4. Return to equations in previous example.

(a) Find slope of tangent line to the graph of $xy = 1$ at $(1, 1)$.

(b) Find equation of tangent line to the graph of $x^2 + y^2 = 1$ at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Notice that it does not make sense to ask for the tangent line at $x = \frac{\sqrt{2}}{2}$. Why?

(c) Have fun solving for y in $x^3+y^3 = 3xy$. The naive approach of solving for y and then taking derivatives will *not* work here.

Important Note 5. Given an implicit function determined by an equation in x and y, we cannot always solve for y and then take the derivative. So, we need a better technique.

Implicit differentiation

If y is implicitly a function of x (peek under y and see x's running around), we can use the chain rule to differentiate.

Example 6. Assume that y is a differentiable function of x.

(a) $\frac{d}{dx}\left[x^2\right] =$ ____________

(b) $\frac{d}{dx}\left[y^2\right] =$ ____________

Example 7. Find $\frac{dy}{dx}$ at the specified point.

(a) $xy = 1$, $(1, 1)$

(b) $x^2 + y^2 = 1$, $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

(c) $x^3 + y^3 = 3xy$, $(3/2, 3/2)$

Note 8. After taking the derivative of both sides of the equation with respect to x, the steps for solving for $\frac{dy}{dx}$ are always roughly the same.

1. Multiply everything (or as much as you need to) out.
2. Get all terms with a factor of $\frac{dy}{dx}$ on one side of the equation and all other terms on the other side of the equation.
3. Factor out $\frac{dy}{dx}$.
4. Divide to obtain $\frac{dy}{dx}$

Let's do one last example.

Example 9. Find $\frac{dy}{dx}$ for $e^x \ln y + \cos x = \sin y$.