Section 3.4: The Quotient Rule

Goal

In this section, we will introduce a shortcut for finding derivatives of quotients.

Review

Recall our formal definition of the derivative. If f is a function, then the derivative of f is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

whenever this limit exists (and it may not at some values).

So far we have encountered each of the following shortcuts, each of which follows from the limit definition of the derivative:

(a) $\boxed{\frac{d}{dx} x^n = \underline{\hspace{3cm}}}$ (power rule)

(b) $\boxed{\frac{d}{dx}\left(f(x) + g(x)\right) = \underline{\hspace{3cm}}}$ (sum rule)

(c) $\boxed{\frac{d}{dx}\left(cf(x)\right) = \underline{\hspace{3cm}}}$ (constant multiple rule)

(d) $\boxed{\frac{d}{dx}\left(f(x)g(x)\right) = \underline{\hspace{6cm}}}$ (product rule)

Initial discussion

Our goal is to be able to easily compute the derivative of the quotient of two differentiable functions. However, notice that

$$\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)},$$

which means that we can always interpret a quotient as a product. Yet, in this case, this isn't very helpful for us since we don't have a way to compute

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right).$$

Let's focus on $\frac{1}{g(x)}$ for a moment.

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We see that

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{1}{g(x)}\right) \\
&= \lim_{\Delta x \to 0} \frac{\frac{1}{g(x+\Delta x)} - \frac{1}{g(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{g(x)-g(x+\Delta x)}{g(x+\Delta x)g(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{g(x) - g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x} \\
&= \lim_{\Delta x \to 0} -\frac{g(x+\Delta x) - g(x)}{\Delta x} \cdot \frac{1}{g(x+\Delta x)g(x)} \\
&= -\frac{g'(x)}{g(x)^2}
\end{aligned}$$

That is, $\boxed{\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{g(x)^2}}$.

Using this formula, we can sort out the quotient rule.

The Quotient Rule

The following theorem shows us how to take the derivative of a quotient.

Theorem 1 (The quotient rule)**.** If f and g are differentiable functions, then

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \underline{\hspace{6cm}}.}$$

Proof. We see that

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \\
&= \underline{\hspace{5cm}} \\
&= \underline{\hspace{5cm}} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
\end{aligned}$$

□

Let's do some examples.

Examples

Example 2. Compute the derivative of $f(x) = \dfrac{5x-3}{\sqrt{x}}$ in *three different ways.*

Example 3. Compute the derivative of each of the following functions.

(a) $f(x) = \dfrac{4-x}{3x^2+7x-3}$

(b) $g(x) = \dfrac{1}{4-9x^2}$

(c) $f(x) = \dfrac{\sqrt{625 - x^2}}{x^2 - 2x + 5}$ *Hint:* $\dfrac{d}{dx}\sqrt{625 - x^2} = \dfrac{-x}{\sqrt{625 - x^2}}$

Example 4. Find the equation of the tangent line to the graph of $f(x) = \dfrac{x^{3/2}}{x^2 + 1}$ when $x = 1$.

