# Section 3.3: The Product Rule

## Goal

In this section, we will introduce a shortcut for finding derivatives of products.

## Review

Recall our formal definition of the derivative. If $f$ is a function, then the derivative of $f$ is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

whenever this limit exists (and it may not at some values).

So far we have encountered each of the following shortcuts, each of which follows from the limit definition of the derivative:

(a) $\frac{d}{dx} x^n =$ ____________ (power rule)

(b) $\frac{d}{dx}(f(x) + g(x)) =$ ____________ (sum rule)

(c) $\frac{d}{dx}(cf(x)) =$ ____________ (constant multiple rule)

## Motivating example

Our goal is to be able to easily compute the derivative of products of two (or more) differentiable functions. Since the derivative of a sum of functions is just the sum of the derivatives (that's (b) above), one might be inclined to think that we can just take the derivative of a product by taking the product of the derivatives. That is, maybe you think

$$\frac{d}{dx}(f(x)g(x)) \stackrel{?}{=} f'(x)g'(x).$$

Well, let's give this a whirl with the function $f(x) = (1 - x)(2x^2 - 3x + 1)$ and see what happens. We see that

$$\frac{d}{dx}(1 - x) \cdot \frac{d}{dx}(2x^2 - 3x + 1) = \underline{\hspace{6cm}} = \underline{\hspace{3cm}}.$$

The good news is that we can check to see if this is correct by just multiplying out the original function and then taking the derivative of the resulting expression using linearity of the derivative and the power rule. Let's do that now. First, we see that

$$f(x) = (1 - x)(2x^2 - 3x + 1) = \underline{\hspace{6cm}} = \underline{\hspace{3cm}},$$

which implies that

$$f'(x) = \underline{\hspace{6cm}}.$$

We can conclude that ____________ $\neq$ ____________, and hence ________________________.

Okay, so the derivative of a product is not necessarily equal to the product of the derivatives. What is the appropriate formula?

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# The Product Rule

The following theorem shows us how to take the derivative of a product.

**Theorem 1** (The product rule)**.** If $f$ and $g$ are differentiable functions, then

$$\frac{d}{dx}(f(x)g(x)) = \underline{\hspace{4cm}}.$$

*Proof.* We see that

$$\begin{aligned}
&\frac{d}{dx}\left(f(x)g(x)\right) \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} && \text{(definition of derivative)} \\
&= \lim_{\Delta x \to 0} \underline{\hspace{6cm}} && \text{(add/subtract } f(x+\Delta x)g(x)\text{)} \\
&= \lim_{\Delta x \to 0} \underline{\hspace{4cm}} + \lim_{\Delta x \to 0} \underline{\hspace{4cm}} && \text{(split into two quotients)} \\
&= \lim_{\Delta x \to 0} \underline{\hspace{4cm}} + \lim_{\Delta x \to 0} \underline{\hspace{4cm}} && \text{(factor)} \\
&= f'(x)g(x) + f(x)g'(x),
\end{aligned}$$

where in the second to last step we used the definition of the derivative for both $f$ and $g$ and the fact that if $f$ is differentiable, then $f$ is continuous, which implies that $\lim_{\Delta x \to 0} f(x+\Delta x) = f(x)$. □

**Note 2.** Instead of using $f$ and $g$, we should remember the product rule as:

> The derivative of a product is equal to the derivative of the first function times the second function plus the first function times the derivative of the second function.

Or, shorter:

$$(\text{first} \cdot \text{second})' = (\text{first})' \cdot (\text{second}) + (\text{first}) \cdot (\text{second})'.$$

**Note 3.** Does the product rule agree with the constant multiple rule? Well, let's check. Let $f$ be a differentiable function and let $c$ be a constant. Using the product rule, we see that

$$\frac{d}{dx}(cf(x)) = \underline{\hspace{4cm}} = \underline{\hspace{2cm}},$$

which agrees with the constant multiple rule.

Let's do some examples.

## Examples

**Example 4.** Compute the derivative in *two different ways* of each of the following functions.

(a) $f(x) = (6x^2 + x)(3 - 5x)$

 

(b) $g(x) = \dfrac{x+4}{\sqrt{x}}$

**Example 5.** Compute the derivative of each of the following functions.

(a) $f(x) = (2x^3 - 4x^2 + 5x - 7)(\sqrt{x} - x^{3/2} + 17)$

(b) $y = (4 - 3x^5)^2$

**Example 6.** Using the fact that $\dfrac{d}{dx}\sqrt{625 - x^2} = \dfrac{-x}{\sqrt{625 - x^2}}$ (*note:* you will need to use this fact on your homework), find the equation of the tangent line to the graph of $f(x) = x^2\sqrt{625 - x^2}$ when $x = 1$.

 