Section 3.2: Linearity of the Derivative

Goal

In this section, we will introduce our second shortcut for finding derivatives that allows us to handle multiplying by constants and adding/subtracting functions.

Review

Recall our formal definition of the derivative. If f is a function, then the derivative of f is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

whenever this limit exists (and it may not at some values).

As we mentioned last time, using the limit definition can be cumbersome, and so we desire some shortcuts for finding derivatives. So far we have one shortcut, which is called the *power rule*:

$$\boxed{\frac{d}{dx} x^n = \underline{\hspace{3cm}}}$$

This allows us to compute the derivative of any power of x. Our immediate desire is to find a way to easily compute the derivative of functions like $f(x) = 3/x^2 - 5x^3 + 4x - 3$. Fortunately, this isn't too hard!

General linearity

We say that a function is *linear* if it "respects addition and multiplication." More specifically, a function f is linear iff

1. $f(x + y) = \underline{\hspace{3cm}}$,
2. $f(cx) = \underline{\hspace{3cm}}$ for all constants c.

As you might guess, being linear has something to do with lines. However, perhaps surprisingly, not every line is linear. In fact, most functions are not linear!

Let $f(x) = mx$, where m is a fixed constant. Observe that

$$f(x + y) = \underline{\hspace{3cm}} = \underline{\hspace{3cm}} = \underline{\hspace{3cm}},$$

and for all constants c, we see that

$$f(cx) = \underline{\hspace{3cm}} = \underline{\hspace{3cm}} = \underline{\hspace{3cm}}.$$

Thus, the function $f(x) = mx$ is linear. However, if $f(x) = mx + b$, where both m and b are fixed constants with $b \neq \underline{\hspace{3cm}}$, then we have

$$f(x + y) = \underline{\hspace{3cm}} = \underline{\hspace{3cm}}.$$

On the other hand, we see that

$$f(x) + f(y) = \underline{\hspace{3cm}} = \underline{\hspace{3cm}}.$$

Therefore, $f(x) = mx + b$ (with $b \neq 0$) is *not* linear.

Here are a few more examples of functions that are *not* linear.

(a) $\sqrt{cx} \neq c\sqrt{x}$, as long $c \neq 0$

(b) $(x+y)^2 \neq x^2 + y^2$

We say that an operation is linear if it respects addition and multiplication. The good news is that the derivative is a linear operation!

Linearity of the derivative

Theorem 1 (Linearity of the derivative)**.** If f and g are differentiable functions and c is a constant, then

(a) $\boxed{\frac{d}{dx}\left(f(x)+g(x)\right) = \underline{\hspace{3cm}}}$ (sum rule)

(b) $\boxed{\frac{d}{dx}\left(cf(x)\right) = \underline{\hspace{3cm}}}$ (constant multiple rule)

Proof of part (a). We see that

$$\begin{aligned}
\frac{d}{dx}\left(f(x)+g(x)\right) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)+g(x+\Delta x)-(f(x)+g(x))}{\Delta x} && \text{(definition of derivative)}\\
&= \lim_{\Delta x \to 0} \underline{\hspace{9cm}} && \text{(distribute minus sign)}\\
&= \lim_{\Delta x \to 0} \underline{\hspace{9cm}} && \text{(rearrange terms)}\\
&= \lim_{\Delta x \to 0} \underline{\hspace{9cm}} && \text{(split into two quotients)}\\
&= \lim_{\Delta x \to 0} \underline{\hspace{9cm}} && \text{(distribute limit)}\\
&= f'(x) + g'(x).
\end{aligned}$$

□

Geometric intuition about part (b). If a function f is differentiable, then its graph is "smooth" with no vertical tangents, and is hence continuous. To simplify the discussion, assume that $c > 0$ (the case with $c = 0$ is easy and the case with $c < 0$ can be handled in a similar manner to when $c > 0$). How do the graphs of $y = f(x)$ and $y = cf(x)$ compare?

How do the slopes of the tangent lines to $y = f(x)$ compare to the slopes of the tangent lines to $y = cf(x)$?

□

Using the two shortcuts of the theorem, we can quickly compute functions which are sums of constants times powers of x.

Let's do some examples.

Examples

Example 2. Compute the derivative of each of the following functions.

(a) $f(x) = 6x^2$

(b) $g(x) = 2x^2 - 5x + 1$

(c) $h(x) = 3\sqrt{x} - \dfrac{2}{\sqrt{x}} + \sqrt{2}$

(d) $y = \dfrac{x^3}{4} - 2x^{3/2} + \pi^2$

(e) $y = (x-1)(2x+3)$

Example 3. Find the equation of the tangent line to the graph of $f(x) = x - 3x^2$ when $x = 1$.