Section 3.1: The Power Rule

Goal

In this section, we will introduce one of our first shortcuts for finding derivatives.

Motivation

Recall our formal definition of the derivative.

Definition 1. Let f be a function. Then the derivative of f is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

whenever this limit exists (and it may not at some values).

Also, recall this very important fact.

Important Note 2. The derivative gives a formula for __________________________________. In particular, $f'(a)$ is the __________________________________.

Given any function that has a derivative, if we are clever enough (and patient enough!), we can find the derivative using the limit definition. However, as we've seen, this can be extremely tedious. We need some shortcuts!

Before introducing any shortcuts, it is important to emphasize that the limit definition (which has a nice picture to go with it) is always running the show. Any attempt to understand the concept of derivatives fully must eventually appeal to the limit definition. You can "do" calculus without limits, but you cannot really understand why things work without limits.

The Power Rule

What we would like to have is a rule for computing

$$\frac{d}{dx} x^n$$

for any n. After computing the derivative of a few functions of the form $f(x) = x^n$ using the definition of the derivative, you might be able to conjecture that the following is true.

Theorem 3 (Power Rule)**.**

$$\boxed{\frac{d}{dx} x^n = \underline{\hspace{3cm}}}$$

It is easy to say that this is true, but why is it true?

Sketch of proof of special case with n a positive integer. We see that

$$\frac{d}{dx} x^n = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}.$$

To get any traction here, we need to know what $(x + \Delta x)^n$ equals, but we don't want to over do it. All we really need to know is what the terms are with the two highest power of x and that everything else has at least one factor of Δx. It isn't too difficult to see that

$$(x + \Delta x)^n = x^n + \boxed{n} x^{n-1} \Delta x + (\text{lots of other terms with at least two factors of } \Delta x).$$

The only real difficulty above is understanding why the coefficient on the second term is n. One thing that might help convince you of this is to compute a bunch of expansions using something like `Sage`. If you use the following code in a `Sage` cell, you will generate ten expansions of $(x+y)^n$:

```
var('x,y')
for n in [1..10]:
     expand((x+y)^n)
```

Okay, what we can now conclude is that

$$\begin{aligned}\frac{d}{dx}x^n &= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + (\text{lots of other terms with at least two factors of } \Delta x) - x^n}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + (\text{lots of other terms with at least two factors of } \Delta x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} nx^{n-1} + (\text{lots of other terms with at least one factor of } \Delta x) \\ &= nx^{n-1},\end{aligned}$$

as expected. □

The moral of the story is that we can now apply the power rule whenever we want to take the derivative of x to a power.

Examples

Compute the derivative of each of the following functions.

(a) $f(x) = x^7$

(b) $g(x) = x^{-3}$

(c) $h(x) = \sqrt{x}$

(d) $y = \dfrac{1}{x^{3/5}}$

(e) $y = \dfrac{1}{\sqrt[3]{x}}$

