Section 2.3: Limits (part 2)

Goal

In this section, we will develop some tools for working with limits algebraically.

More on epsilonics

Recall our formal definition of the limit.

Definition 1. Let f be a function defined on some open interval that contains the number a, except possibly a itself. Then we say that the *limit of $f(x)$ as x approaches a is L*, and we write

$$\lim_{x\to a} f(x) = L$$

if for every number $\epsilon > 0$, there is a number $\delta > 0$ such that

$$|f(x) - L| < \underline{\hspace{3cm}}$$

whenever

$$0 < |x - a| < \underline{\hspace{3cm}}.$$

Let's briefly fiddle around with the applet that we saw last time to remind ourselves how this works:

http://dcernst-teaching.wikidot.com/notes:epsilon-delta-limit

As we saw at the end of the last set of notes, using the formal definition in even a "simple" example (where it is completely obvious what the answer is) can be cumbersome. We need some tools to help us out!

Limit laws

Here is a collection of facts that all follow from the formal definition of the limit.

Theorem 2. Suppose that $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, k is any constant, and n is a positive integer. Then

1. $\lim_{x\to a} k = \underline{\hspace{3cm}}$
2. $\lim_{x\to a} x = \underline{\hspace{3cm}}$
3. $\lim_{x\to a} f(x)g(x) = \underline{\hspace{3cm}}$ (Theorem 2.6)
4. $\lim_{x\to a} kf(x) = \underline{\hspace{3cm}}$ (Theorem 2.7)
5. $\lim_{x\to a} (f(x) \pm g(x)) = \underline{\hspace{3cm}}$ (Theorem 2.7)
6. $\lim_{x\to a} x^n = \underline{\hspace{3cm}}$
7. $\lim_{x\to a} \frac{f(x)}{g(x)} = \underline{\hspace{3cm}}$, as long as $\underline{\hspace{3cm}}$ (Theorem 2.7)

Selected proofs.

1. Let $\epsilon > 0$. Then for any dang δ, we will always have $|k - k| < \epsilon$. This shows that $\lim_{x\to a} k = k$.

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2. Let $\epsilon > 0$. Choose $\delta =$ ____________ and assume that $|x - a| < \delta$. Then
$$|f(x) - a| = |x - a| < \delta\_\_ \text{ \_\_\_\_\_\_\_\_\_\_\_\_},$$
which shows that $\lim_{x\to a} x = a$.

3. See proof of Theorem 2.6 in book.

4. This follows immediately from 1 and 3.

6. This follows after making repeated applications of 3.

□

Here are a few more tools for our tool box.

Theorem 3 (Theorem 2.10)**.** Suppose that $\lim\limits_{x\to a} g(x) = L$ and $\lim\limits_{x\to L} = f(L)$. Then
$$\lim_{x\to a} f(g(x)) = \text{\_\_\_\_\_\_\_\_\_\_\_\_}.$$

This one should really be in the book, but isn't!!!

Theorem 4. If f and g are two functions that agree everywhere except at possibly a single point, then
$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

Theorem 5 (Theorem 2.11)**.** Suppose that n is a positive integer. Then
$$\lim_{x\to a} x^{1/n} = \text{\_\_\_\_\_\_\_\_\_\_\_\_},$$
provided a is positive when n is ____________.

The big picture

Important Note 6. The upshot of all these theorems is that we can now use them as tools that allow us to compute limits without having to appeal to the definition. Instead we can mostly rely on algebraic manipulations.

Note 7. Here is an algorithm to consider when attempting to evaluate most of the limits we will see for a while. It gets more complicated later. It is important to note that I'm not suggesting that this is work you should necessarily show, but rather this is a thought process for attacking limit problems.

1. Plug in a for x and see what happens.

2. If nothing weird happens at $x = a$ (no holes, no gaps, no asymptotes, etc.), then you can evaluate the limit by just evaluating the function. (The special name for when this happens is *continuous*, which we will talk about later.)

3. If something weird is happening, then we need to do more work. In particular, if after plugging in a for x you have:

   (a) The form $\frac{0}{0}$, then try to (i) factor and cancel, (ii) multiply by the conjugate, or (iii) get rid of complex fractions. Your goal is to get rid of the $\frac{0}{0}$ problem. Theorem 4 is playing a crucial role here.

   (b) The form $\frac{\#}{0}$, where $\# \neq 0$, then the limit does not exist (DNE). Later, we might be able specify whether the limit is "shooting off" to ∞ or $-\infty$ or simply does not exist.

Examples

Example 8. Compute the following limits.

1. $\lim_{x \to 2} \frac{x^2 - 4}{x^2 + x - 2}$

2. $\lim_{x \to 2} \frac{x^2 - 4}{x^2 - x - 2}$

3. $\lim_{x \to 1} \frac{x^2 - 4}{x^2 + x - 2}$

4. $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$

5. $\lim_{x \to 1} \frac{\frac{1}{x+3} - \frac{1}{4}}{x - 1}$