Section 2.3: Limits (part 1)

Goal

In this section, we will introduce the precise definition of a limit and develop some intuition of this technical definition by looking at examples and an applet.

Background

Let f be a function. The slope of the chord between $[x, x + \Delta x]$ is given by

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

For a concrete x-value and a concrete Δx, the slope of the chord will be a number.

Finding the slope of the tangent line to f at x is much more difficult (since finding the slope of a line requires two points). Our way around this difficulty was to see what happens as "Δx approaches 0," which "slides" the chords "closer" to the tangent line. Algebraically, this amounted to doing some simplification of the difference quotient (i.e. whack the Δx in the denominator) and then just plugging in 0 for Δx.

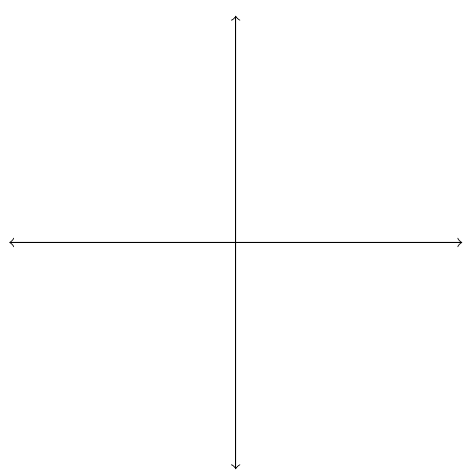
This idea of letting Δx get "arbitrarily close to 0" is a little vague and our immediate goal is make this concept a little more rigorous. In particular, we need a precise way of taking about "arbitrarily close."
For now, let's put the difference quotient away. We'll come back to it after we tackle limits.

A motivating example

Let's consider the function

$$f(x) = \begin{cases} 3x + 1, & x \neq 2 \\ 5, & x = 2. \end{cases}$$

The graph of this function looks like:



Intuitively, we see that when x is close to 2 but $x \neq 2$, $f(x)$ is close to ____________. Soon we will say something like "the limit of $f(x)$ as x approaches 2 is ____________" and write

$$\lim_{x \to 2} f(x) = \text{\_\_\_\_\_\_\_\_\_\_\_\_}.$$

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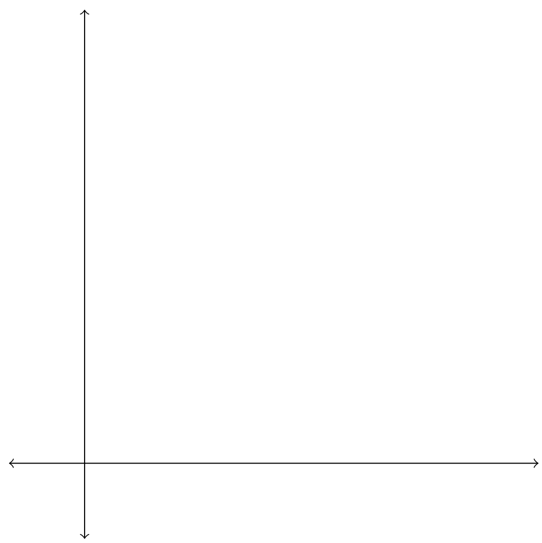
Let's explore what "close" means here. How close to 2 does x have to be so that $f(x)$ differs from ____________ by less than 0.1?

Recall that the distance from x to 2 is ____________ and the distance from $f(x)$ to 7 is ____________. Our goal is to find a distance, which we will call δ, such that

$$\_\_\_\_\_\_\_\_\_\_ < 0.1 \text{ whenever } 0 < \_\_\_\_\_\_\_\_\_\_ < \delta.$$

Why did we require $0 < |x - 2|$? Because we want to know what happens "near" $x = 2$, not at $x = 2$.

Here's the corresponding picture



Let's fiddle around and see what we can find out:

What we just learned was that we should take $\delta =$ ____________. That is, if x is within a distance of ____________ from 2, then $f(x)$ will be within a distance of 0.1 from 7.

Now, if we change 0.1 to a smaller number like 0.001, then by using a similar argument we could find the corresponding δ (which will be ____________). For 7 to be the precise limit of $f(x)$ as x approaches 2, we must not only be able to bring the difference between $f(x)$ and 7 below 0.1 and 0.001, but we must be able to bring it below *any* positive number.

Epsilonics

Definition 1. Let f be a function defined on some open interval that contains the number a, except possibly a itself. Then we say that the *limit of* $f(x)$ *as* x *approaches* a *is* L, and we write

$$\lim_{x \to a} f(x) = L$$

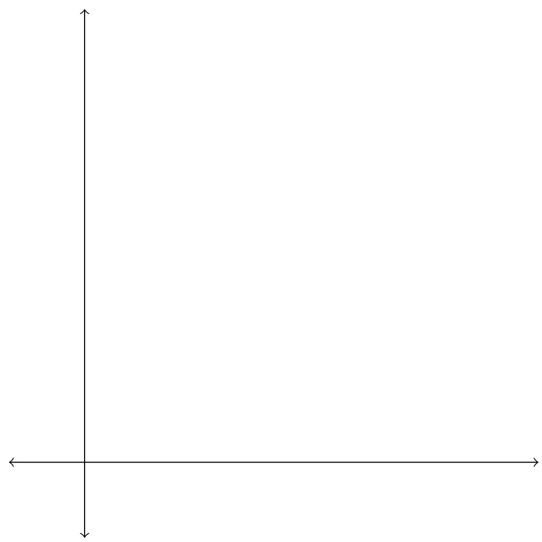
if for every number $\epsilon > 0$, there is a number $\delta > 0$ such that

$$|f(x) - L| < \underline{\hspace{3cm}}$$

whenever

$$0 < |x - a| < \underline{\hspace{3cm}}.$$

Here's the picture:



Here is the intuitive definition of the limit that the above definition is trying to capture.

Intuitive Definition 1'. The limit of $f(x)$ as x approaches a is L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (on either side) but not equal to a.

Let's do some exploring with the applet found at

`http://dcernst-teaching.wikidot.com/notes:epsilon-delta-limit`.

An analogy

We'll think of the $\epsilon - \delta$ definition of the limit as a card game. The game is between a dealer, let's say Plato, and you. There are many hands in the game. There are two ways to win this game, but for now, we'll just talk about one way to win.

In the first way to win you must win every hand. When the game starts, Plato hands you a function and an x-value a. You then pick L. Each hand begins by the dealer handing you an $\epsilon > 0$. This determines an ϵ-tube of y-values, where the tube is perpendicular to the y-axis. (These are the eligible targets in our missile analogy.)

If you can find a $\delta > 0$ such that the part of the function sitting inside the intersection of Plato's ϵ-tube and your δ-tube goes out the sides of the corresponding rectangle (and not the top and bottom; the corners are fine), then you win the hand.

You win the whole game if you can win every single hand. That is, you must win each hand for every possible ϵ.

Examples

Let's do an example where we try to win just a single hand.

Example 2. Let $f(x) = 4x + 1$ and $\epsilon = 0.1$. Find $\delta > 0$ such that $|f(x) - 5| < \epsilon$ whenever $|x - 1| < \delta$.

Now, let's try an example where we win all the hands of the game.

Example 3. Using the formal definition of a limit, prove that $\lim_{x\to 1} 2x + 3 = 5$.

As you can see, even in what would appear to be a simple example, using the formal definition can be tedious. And to make matters worse, things can get worse quite quickly. The good news is that using our definition, we can prove some general theorems that will make computing limits *much* easier!

